Int. J. Multiphase Flow Vol. 14, No. 3, pp. 253–264, 1988
Printed in Great Britain.
0301-9322/88 $3.00 + 0.00

INTERPHASE MOMENTUM INTERACTION EFFECTS IN THE AVERAGED MULTIFIELD MODEL

PART II: KINEMATIC WAVES AND INTERFACIAL DRAG IN BUBBLY FLOWS

C. PAUCHON† and S. BANERJEE
Department of Chemical and Nuclear Engineering, University of California, Santa Barbara, CA 93106, U.S.A.

(*Received* 7 *May* 1987; *in revised form* 30 *December* 1987)

Abstract—The model proposed in Part I [*Int. J. Multiphase Flow* **12,** 559–573 (1986)] for bubbly flows is extended to include the effect of velocity distributions around bubbles and the Riemann invariants are calculated to demonstrate that the void fraction is the conserved quantity that propagates along the faster characteristics. It is shown that kinematic wave velocities based on a constant interfacial friction coefficient propagate with velocities close to but slightly greater than that for the faster characteristic and are weakly unstable. Neutral stability of kinematic waves is found to imply a form of the interfacial friction coefficient in remarkable agreement with experiment. Wall friction has a negligible effect on these results for the usual range of parameters. Bubble interactions as described by a bubble-in-cell model do not affect the results up to a void fraction of 0.1. Beyond these void fractions the cell model predicts significant effects but experimental data suggest that the effects may be overestimated. Turbulence is shown to provide axial dispersion of void fraction which stabilizes the system of bubbly flow equations.

1. INTRODUCTION

In Part I of this series (Pauchon & Banerjee 1986), void propagation was studied within the framework of the multifield model. [For discussion of the general model, see Delhaye (1968), Ishii (1975), Yadigaroglu & Lahey (1976), Banerjee & Chan (1980) and Drew (1983), amongst many others.] Pauchon & Banerjee (1986) showed that experimental data on void propagation were reasonably well predicted by the faster characteristic velocity if care was taken to model interphase pressure interactions, which lead to the virtual mass term and a term involving the spatial gradient of the void fraction. As discussed later, if there are no pressure variations over the interface then the virtual mass effect and/or form drag are absent. These terms have important effects on the mathematical structure of the system and, in particular, on the characteristic velocities [see also Ramshaw & Trapp (1978), Drew (1983) and Jones & Prosperetti (1985)].

Interesting issues left unresolved but arising from Part I are the role of kinematic waves in void propagation and the relationship between kinematic waves and the characteristics. These are the main problems tackled in this paper, with some additional attention being paid to extension of the work in Part I, to include the effects of velocity distribution around bubbles and bubble interactions. In addition the effect of turbulence on void propagation is considered.

In the following sections, we will first recapitulate the space/ensemble (or time) averaged multifield formulation and extend the results of Part I to include velocity distributions around bubbles. The characteristics will be derived together with the Riemann invariants to demonstrate that the void fraction is indeed the conserved quantity propagating along the faster characteristic. This will be followed by consideration of kinematic waves and their stability—first assuming constant interfacial drag. The relationship between characteristics and kinematic waves to ensure stability will then be examined and the implications on the form of the interfacial drag relationship elucidated. Finally, the effects of bubble interactions and turbulence will be discussed.

†Present address: SINTEF, Trondheim, Norway.

2. THE MULTIFIELD MODEL FOR BUBBLY FLOW

2.1. The linear momentum equation and closure relationships

The linear momentum equation for field k has been derived by Banerjee & Chan (1980), among others, as

$$\frac{\partial}{\partial t}\epsilon_k\langle\rho_k u_k\rangle+\frac{\partial}{\partial z}\epsilon_k\langle\rho_k u_k^2\rangle+\epsilon_k\frac{\partial p_k}{\partial z}-\frac{\partial}{\partial z}\langle\mathbf{n}_z\cdot(\boldsymbol{\tau}_k\cdot\mathbf{n}_z)\rangle$$
$$=\Delta p_{ki}\frac{\partial\epsilon_k}{\partial z}+\langle\epsilon_k\rho_k F_k\rangle-\langle m_k u_k\rangle_i-\langle\mathbf{n}_k\cdot\mathbf{n}_z\Delta p'_{ki}\rangle_i+\langle\mathbf{n}_k\cdot\mathbf{n}_z\cdot\boldsymbol{\tau}_k\rangle_i+\langle\mathbf{n}_{kw}\cdot\mathbf{n}_z\cdot\boldsymbol{\tau}_k\rangle_w,\quad[1]$$

where

$$\langle f_k\rangle=\frac{1}{V_k}\int_{v_k}f_k\,\mathrm{d}v$$

and

$$\langle f_k\rangle_i=\frac{1}{V}\int_{a_i}f_k\,\mathrm{d}s,$$

ϵ_k is the volume fraction of field k in the volume V, $\mathbf{n}_k$ is the outward drawn normal on the surface of field k, $\mathbf{n}_z$ is the unit vector in the z-direction, a_i is the interfacial area of field k and a_{kw} the area of contact between field k and the wall. (See also figure 1 in which the symbols are defined.) The other variables are: ρ_k, the density; u_k, the velocity in the z-direction; p_k, the pressure; $\boldsymbol{\tau}_k$, the shear stress tensor; F_k, the body force; and $\dot{m}_k$, the mass transfer out of field k. Several interesting aspects are apparent in [1]. The pressure at the interface p_{ki} has been broken into a part that varies over the interface and parts that do not:

$$p_{ki}=\langle p_k\rangle+\Delta p_{ki}+\Delta p'_{ki}\quad[2]$$

with

$$\Delta p_{ki}=\frac{V}{a_i}\langle p_{ki}\rangle_i-\langle p_k\rangle,$$

which does not vary over a_i in V, and

$$\Delta p'_{ki}=p_{ki}-\frac{V}{a_i}\langle p_{ki}\rangle_i.$$

Note that $\langle p_{ki}\rangle_i$ is dimensionally not a pressure since it is the pressure over all the interface per unit volume. Δp_{ki} is the difference between the average pressure at the interface within phase k and the average pressure in field k. Pauchon & Banerjee (1986) showed that for bubbly flows this results in a term involving a spatial gradient of void fraction, which has a significant effect on the nature

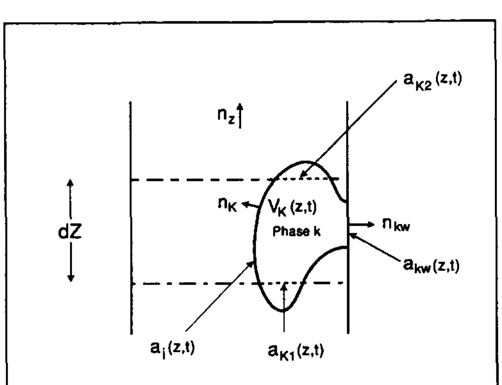

Figure 1. Schematic of two-phase flow defining the symbols.

of the characteristics. Based on potential flow theory around a sphere of constant radius, it was established that

$$\Delta p_{\mathrm{Li}} = -\xi \rho_{\mathrm{L}} \langle (u_{\mathrm{G}} - u_{\mathrm{L}}) \rangle^2, \qquad \xi = \tfrac{1}{4}, \tag{3}$$

while $\Delta p_{\mathrm{Gi}} = 0$. Another important term is $\langle \mathbf{n}_k \cdot \mathbf{n}_z \Delta p'_{ki} \rangle_{\mathrm{i}}$, which is the force per unit volume in the z-direction, due to pressure variations over a_{i}. For accelerating flows, this term is significant even if the phases are considered inviscid. The term is generally written as the product of the dispersed phase volume fraction (here the gas) × the continuous phase density (here the liquid) × the so-called virtual mass acceleration. In the limit of a single infinitesimal sphere, Voinov (1973) established that

$$\langle \mathbf{n}_k \cdot \mathbf{n}_z \Delta p'_{ki} \rangle_{\mathrm{i}} = \pm \alpha_{\mathrm{G}} \rho_{\mathrm{L}} C_{\mathrm{VM}} \left(\frac{\mathrm{D_G} u_{\mathrm{G}}}{\mathrm{D}t} - \frac{\mathrm{D_L} u_{\mathrm{L}}}{\mathrm{D}t} \right), \qquad C_{\mathrm{VM}} = \tfrac{1}{2}, \tag{4}$$

where

$$\frac{\mathrm{D}_k}{\mathrm{D}t} = \frac{\partial}{\partial t} + u_k \frac{\partial}{\partial z}.$$

Note that Ruggles *et al.* (1986) showed that $C_{\mathrm{VM}} \approx 1/2$, on the basis of experiment, for $\alpha \lesssim 0.15$ and rises relatively slowly at higher values.

Distribution effects around a single bubble can also be taken into account using potential flow theory. They arise from the fact that in the liquid phase

$$\langle u_{\mathrm{L}}^2 \rangle \neq \langle u_{\mathrm{L}} \rangle^2$$

Hence, the second term on the l.h.s. of [1] has to be broken up into

$$\langle u_{\mathrm{L}}^2 \rangle = \langle u_{\mathrm{L}} \rangle^2 + \langle u_{\mathrm{L}}'^2 \rangle,$$

where u'_{L} is the local perturbation to the volume-averaged liquid velocity due to the presence of a single sphere. Biesheuvel & van Wijngaarden (1984) showed that

$$\langle u_{\mathrm{L}}'^2 \rangle = k \epsilon_{\mathrm{G}} \langle (u_{\mathrm{G}} - u_{\mathrm{L}}) \rangle^2 \quad \text{with} \quad k \cong \tfrac{1}{5}. \tag{5}$$

Finally, interfacial and wall shear stress can be modelled in the usual way:

$$\langle \mathbf{n}_k \cdot \boldsymbol{\tau}_z \rangle_{\mathrm{i}} = \epsilon_{\mathrm{G}} \rho_{\mathrm{L}} \frac{f_{\mathrm{i}}}{D} \langle (u_{\mathrm{G}} - u_{\mathrm{L}}) \rangle^2 \tag{6}$$

and

$$\langle \mathbf{n}_{k\mathrm{w}} \cdot \boldsymbol{\tau}_z \rangle_{\mathrm{i}} = \epsilon_k \rho_{\mathrm{L}} \frac{f_{\mathrm{w}}}{D} \langle u_k \rangle^2, \tag{7}$$

where distribution effects are buried in f_{i} and f_{w} but can be made explicit if necessary through distribution coefficients. Note of course that f_{i} can be a function of $u_{\mathrm{G}} - u_{\mathrm{L}}$. D is a length scale chosen as the pipe diameter, f_{i} and f_{w} are friction coefficients. Here f_{i} is related to the drag coefficient C_{D} by

$$f_{\mathrm{i}} = \tfrac{3}{8} C_{\mathrm{D}} \frac{D}{R}, \tag{8}$$

where R is average bubble radius.

Considering no interphase mass transfer and neglecting axial stress ($\boldsymbol{\tau}_{zz,k}$), all closure relationships needed for [3] can then be specified to the level of the approximations discussed (i.e. no bubble interactions are considered for the pressure interaction terms but may be captured to some extent in f_{i} and f_{w}).

2.2. Assumptions and simplified form

To proceed, the system of bubbly flow equations is simplified while retaining the essence of the problem, assuming that

—the two phases are incompressible
—gas inertia is negligible
—the average gas pressure is equal to the average interfacial pressure and the bubbles are large enough that surface tension forces are small (though these can be incorporated quite easily in the model)
—bubble interactions are negligible
—the two phases are flowing cocurrently with $\langle u_G - u_L \rangle > 0$ and $\langle u_L \rangle > 0$ at all times
—the bubbles remain spherical.

In the following, the averaging signs will be dropped but it is understood that all variables are averaged. A system of six equations are needed in the six unknowns ϵ_G, ϵ_L, u_G, u_L, p_G and p_L. The first three are

$$\frac{\partial \epsilon_G}{\partial t} + u_G \frac{\partial \epsilon_G}{\partial z} + \epsilon_G \frac{\partial u_G}{\partial z} = 0 \qquad [9a]$$

$$\frac{\partial \epsilon_L}{\partial t} + u_L \frac{\partial \epsilon_L}{\partial z} + \epsilon_L \frac{\partial u_L}{\partial z} = 0 \qquad [9b]$$

and

$$\epsilon_G + \epsilon_L = 1. \qquad [9c]$$

From [1]–[7] it follows that the two momentum equations are

$$\epsilon_G \frac{\partial p_G}{\partial z} = -\rho_L \epsilon_G \frac{f_i}{D}(u_G - u_L)^2 - \epsilon_G \rho_L C_{VM}\left(\frac{D_G u_G}{Dt} - \frac{D_L u_L}{Dt}\right) \qquad [9d]$$

$$\epsilon_L \rho_L \frac{D_L u_L}{Dt} + \frac{\partial}{\partial z}\rho_L k \epsilon_G \epsilon_L (u_G - u_L)^2 + \epsilon_L \frac{\partial p_L}{\partial z} = -\xi \rho_L (u_G - u_L)^2 \frac{\partial \epsilon_L}{\partial z} - \epsilon_L \rho_L g$$
$$+ \epsilon_G \rho_L \frac{f_i}{D}(u_G - u_L)^2 + \epsilon_G \rho_L C_{VM}\left(\frac{D_G u_G}{Dt} - \frac{D_L u_L}{Dt}\right) - \epsilon_L \rho_L \frac{f_2}{D} u_L^2 \qquad [9e]$$

with the additional relationship, from [3],

$$p_G - p_L = p_{Li} - p_L = -\xi \rho_L (u_G - u_L)^2 \qquad [9f]$$

(this neglects surface tension); and

$$\xi = \tfrac{1}{4}, \qquad C_{VM} = \tfrac{1}{2}, \qquad k = \tfrac{1}{5}$$

in the limit that there are no bubble interactions. By substituting the two relationships [9c] and [9f] into the four conservation equations, the system is reduced to four equations with four unknowns: $\epsilon_G, u_G, u_L, p_L$. Moreover, assumption of incompressibility of both phases reduces the dimension of the system to 2, corresponding to the fact that the two pressure wave characteristics propagate at infinite velocities.

In terms of u_L and ϵ_L, the model equations in their most condensed form can be written in nondimensional form, neglecting for the moment wall friction, as

$$\frac{\partial \epsilon_L}{\partial t} + \frac{\partial \epsilon_L u_L}{\partial z} = 0 \qquad [10a]$$

and

$$\frac{\partial u_L}{\partial t} + \frac{\partial u_L}{\partial z}[u_L + 2v(u_G - u_L)] + \frac{\partial \epsilon_L}{\partial z}(u_G - u_L)^2\left(\frac{v}{\tau \epsilon_L} - \frac{v^2}{\epsilon_L}\right) = \frac{\epsilon_G}{\tau}[f_i(u_G - u_L)^2 - \epsilon_L \mathrm{Fr}], \qquad [10b]$$

where $u_G - u_L$ can be replaced by

$$u_G - u_L = \frac{j_0 - u_L}{1 - \epsilon_L}, \qquad j_0 = \epsilon_L u_L + \epsilon_G u_G.$$

For a stationary flow problem, j_0 is the constant volumetric flux and

$$v = \epsilon_L^2 \frac{(C_{VM} - \xi - k\epsilon_G)^2}{\epsilon_G \epsilon_L + C_{VM}} + \epsilon_G \epsilon_L (\xi + k - C_{VM}) + 2\epsilon_L^2 \left(\xi - \frac{C_{VM}}{2} \right),$$

$$\upsilon = \epsilon_L \frac{(C_{VM} - \xi - k\epsilon_G)}{\epsilon_G \epsilon_L + C_{VM}},$$

$$\tau = \epsilon_G \epsilon_L + C_{VM}$$

and

$$\mathrm{Fr} = \frac{gD}{(u_G - u_L)^2}.$$

The variables in [10a,b] have been nondimensionalized with $(u_{G0} - u_{L0})$ as a velocity scale and D as a length scale. The index 0 indicates a time- (or ensemble-) averaged value of the variable.

3. CHARACTERISTICS AND KINEMATIC WAVES

The characteristic velocities can be obtained from the derivative terms in [10a,b]. They represent the fastest and slowest wave velocities propagating in the system (Whitham 1974). The quantities conserved along these characteristics are the Riemann invariants which will be determined analytically as functions of α_L and u_L. The nature of the conserved quantities that propagate at the characteristic speeds will then be clarified.

On the other hand, it is expected that density or void fraction propagates at the kinematic wave velocity which depends exclusively on the algebraic terms. These waves are stable if their speed is in between the two characteristic velocities [see also Whitham (1974)]. In the following, we assume that the two dominant algebraic terms are due to interfacial drag and gravity (or buoyancy). The effect of wall drag will be analyzed in a second step together with discussion of bubble interactions.

3.1. Characteristic velocities and Riemann invariants

Omitting algebraic terms, [10a,b] may be case in the form

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{U}}{\partial z} = 0$$

where $\mathbf{U}$ is the vector of dependent variables $(\epsilon_L, u_L)^T$. The eigenvalues of the matrix $\mathbf{A}$ are the characteristic velocities $\lambda^{\pm}$ given in figure 2 as a function of ϵ_{G0} in terms of the normalized quantity

$$\lambda^{*\pm} = \frac{\lambda^{\pm} - u_L}{u_G - u_L} = \upsilon \pm \sqrt{\frac{v}{\tau}}. \qquad [11]$$

From [10a,b] and [11] the model is seen to be hyperbolic for $C_{VM} = k = \xi = 0$. This is as expected, since gas phase inertia has been neglected. It is apparent from the value of v that the virtual mass force (C_{VM}) tends to make the characteristic complex and the system [10a,b] nonhyperbolic. From a physical viewpoint, the destabilizing effect of the virtual mass effect arises because a change in velocity leads to ~ 3 times as large a change in gas velocity. Consequently, when a region suffers a small increase in void fraction, the liquid velocity increases with a consequent larger increase in gas velocity which leads to gas flowing into the region and further increases the void fraction. The reverse is true of the phasic pressure difference coefficient (ξ) and the velocity distribution coefficient (k), which are stabilizing. The quantities propagating at the characteristic velocities are the Riemann invariants $J^{\pm}$, which can be derived analytically for two-dimensional systems. For a detailed account of their derivation and significance see Whitham (1974). In the case under consideration,

$$J^+ = \int \frac{v - \sqrt{\frac{v}{\tau}}}{\epsilon_G \epsilon_L} d\epsilon_L + \int \frac{du_L}{j_0 - u_L} \qquad [12a]$$

and

$$J^- = \int \frac{v + \sqrt{\frac{v}{\tau}}}{\epsilon_G \epsilon_L} d\epsilon_L + \int \frac{du_L}{j_0 - u_L}. \qquad [12b]$$

Notice that j_0 is a constant and that

$$\lambda^{*-} = v - \sqrt{\frac{v}{\tau}} \sim 0$$

over a wide range of void fraction (see the bottom curve in figure 2), so that

$$J^+ \sim -\ln(j_0 - u_L) = -\ln \epsilon_G(u_G - u_L).$$

Note that $u_G - u_L$ is in nondimensional form with $u_{G0} - u_{L0}$ as the nondimensionalizing scale. As the perturbations are small $(u_G - u_L) \sim 1.0$, so $J^+ \sim \ln \epsilon_G$. Thus, the conserved quantity along λ^{*+} is closely approximated by ϵ_G.

3.2. *Stability of kinematic waves*

The previous section indicates that the void fraction propagates along the faster characteristic. On the other hand, it is known [see Wallis (1969)] that the void fraction also propagates at the kinematic wave velocity. This is now evaluated as a function of ϵ_L and u_L. In order to do this, consider algebraic terms of the model system [10a,b] in a frame of reference fixed with the undisturbed liquid and obtain, after linearization:

$$\frac{D_{L0}\epsilon_L}{Dt} + \epsilon_{L0}\frac{\partial u_L}{\partial z} = 0 \qquad [13a]$$

and

$$\frac{D_{L0}u_L}{Dt} + \frac{\partial \epsilon_L}{\partial z}\left(\frac{v_0}{\tau_0 \epsilon_{L0}} - \frac{v_0^2}{\epsilon_{L0}}\right) + \frac{\partial u_L}{\partial z}(2v_0) = \frac{\epsilon_{G0}}{\tau_0}[2f_{i0}(u_G - u_L) - \epsilon_L \mathrm{Fr}_0], \qquad [13b]$$

where the subscript 0 again denotes the unperturbed (stationary) quantities. The kinematic approximation assumes that continuity [13a] is satisfied while τ_0 and v_0 are small and momentum effects reduce to a force balance among the algebraic terms. The wave motion is then governed by

$$\frac{\partial \epsilon_L}{\partial t} + \epsilon_{L0}\frac{\partial u_L}{\partial z} = 0 \qquad [14a]$$

and

$$2f_{i0}(u_G - u_L) - \epsilon_L \mathrm{Fr}_0 = 0, \qquad [14b]$$

and by the condition $j_0 = \epsilon_G u_G + \epsilon_L u_L$, which leads to

$$\epsilon_{G0}(u_G - u_L) + u_L + \epsilon_G = 0. \qquad [14c]$$

Constant interfacial drag. Consider first that if f_i is not a function of the void fraction, then [14a–c] may be written as

$$\frac{\partial \epsilon_L}{\partial t} + \left(\epsilon_{L0} - \frac{\mathrm{Fr}_0}{2f_{i0}}\epsilon_{G0}\epsilon_{L0}\right)\frac{\partial \epsilon_L}{\partial z} = 0. \qquad [15]$$

Clearly, at steady state

$$(u_{G0} - u_{L0})^2 f_{i0} = \epsilon_{L0}\mathrm{Fr}_0, \qquad [16]$$

from which it immediately follows that

$$\frac{\mathrm{Fr}_0}{f_{i0}} = \frac{1}{\epsilon_{L0}}.$$

Thus the kinematic wave velocity a_0^* in a frame of reference fixed with the liquid is

$$a_0^* = \frac{a_0 - u_{L0}}{u_{G0} - u_{L0}} = 1 - \tfrac{3}{2}\epsilon_{G0}. \qquad [17]$$

For the case of constant interfacial drag, figure 3 shows that a_0^* lies above the faster characteristic velocity λ_0^* but close to it. This result is of importance in determining the linear stability of kinematic waves. As we will now show, kinematic waves traveling faster than λ^{*+} are weakly unstable.

Going back to the linearized model [13a,b], we differentiate [13a] with respect to t and [13b] with respect to z and combine to obtain a single equation in terms of ϵ_L:

$$\tau_0\left(\frac{\partial}{\partial t} + v_0\frac{\partial}{\partial z}\right)^2 \epsilon_L - v_0\frac{\partial^2 \epsilon_L}{\partial z^2} = -2f_{i0}\left(\frac{\partial \epsilon_L}{\partial t} + a_0^*\frac{\partial \epsilon_L}{\partial z}\right). \qquad [18]$$

In [18] the term containing v_0 is diffusive, and the separate effects of virtual mass force, pressure difference and velocity distribution around bubbles have already been discussed. We now look at the condition for linear stability of the system. Equation [18] indicates that the global wave motion described on the r.h.s. behaves as

$$\epsilon_L = f(z - a_0^* t),$$

so that

$$\frac{\partial}{\partial t} \sim -a_0^*\frac{\partial}{\partial z}.$$

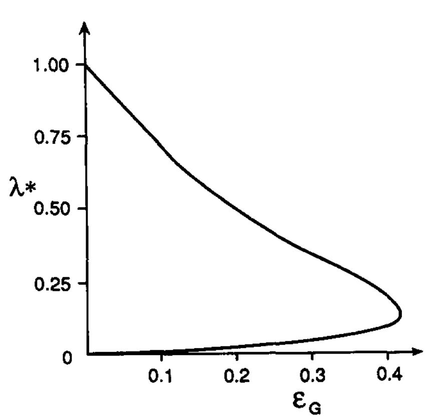

Figure 2. Normalized characteristic speeds as a function of ϵ_G. The model has complex characteristics for $\epsilon_G > 0.42$. $\lambda^* = (\lambda - u_L)/(u_G - u_L)$.

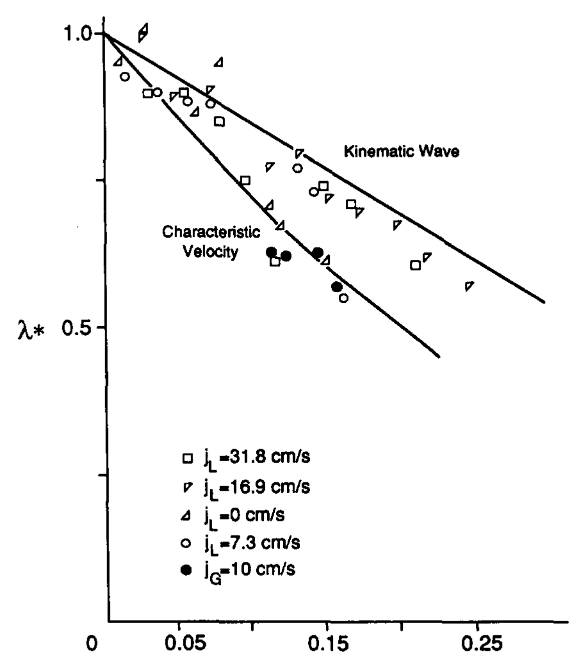

Figure 3. Plots of λ_0^* and a_0^* (straight line) as a function of ϵ_G, together with data points by Bernier (1981) and Pauchon & Banerjee (1986).

In this approximation, we obtain an advection diffusion equation for the void fraction as

$$\frac{\partial \epsilon_L}{\partial t} + a_0^* \frac{\partial \epsilon_L}{\partial z} = -\frac{\tau_0}{2f_{i0}}(\lambda_0^{*+} - a_0^*)(\lambda_0^{*-} - a_0^*)\frac{\partial^2 \epsilon_L}{\partial z^2} \qquad [19]$$

which implies the stability condition

$$\lambda_0^{*-} \leqslant a_0^* \leqslant \lambda_0^{*+}, \qquad [20]$$

since if the coefficient of the diffusive term is negative the flow will be unstable; i.e. the voids will tend to form clumps and coalesce. The kinematic waves predicted with constant interfacial drag have velocities slightly above the two characteristic velocities $\lambda^{*\pm}$, as indicated in figure 3. The experimental measurements actually cluster between the kinematic wave and the faster characteristic velocities, which appears to indicate weakly unstable bubbly flow from [19]. However, as shown later, turbulence would tend to stabilize the system.

Interfacial drag as a function of void fraction. Note that the above result is obtained with a constant interfacial drag coefficient, which implies no bubble interaction effects. While the experimental results of Ruggles *et al.* (1986), alluded to previously, support such a description up to a relatively high void fraction for the virtual mass type terms, the interface drag terms may be more sensitive. To elucidate the situation, we now investigate the form of the interfacial drag coefficient that will give neutrally stable kinematic waves. To proceed, neutral stability for the kinematic waves is obtained when

$$a_0^* = \lambda_0^{*+} \qquad [21]$$

From [15], this implies

$$f_i(\epsilon_{G0}) = \frac{\mathrm{Fr}_0 \epsilon_{G0} \epsilon_{L0}}{2(\epsilon_{L0} - \lambda_0^{*+})}; \qquad [22]$$

or from [8],

$$C_D(\epsilon_{G0}) = \left[\frac{8}{3}\frac{Rg}{(u_{G0} - u_{L0})^2}\right]\frac{\epsilon_{L0}\epsilon_{G0}}{2(\epsilon_{L0} - \lambda_0^{*+})}. \qquad [23]$$

The coefficient in square brackets on the r.h.s. of [23] is the term that arises naturally from a force balance on a single bubble (Ishii & Zuber 1979). The unbracketed term on the far right is the void fraction dependency predicted. We can now compare this expression to that of Ishii & Zuber (1979) in their so-called "Newton's regime". Figure 4 shows agreement up to a void fraction of 0.3. This is well beyond the range of our model as bubble interaction effects would begin to affect the virtual mass, void fraction gradient and velocity distribution terms—all of which we have assumed to be of the highly dilute form. However, on the basis of the calculations presented later, we expect bubble interaction effects to hardly affect the results up to $\alpha_G \sim 0.1$ and perhaps somewhat beyond. Therefore there is a significant range where the analysis is applicable.

The remarkable result here is the agreement of an essentially empirical drag coefficient result with a model which only demands neutral stability of the kinematic waves. Alternatively we may start from a good model for drag and determine its consequences.

A model for the drag coefficient accounting for turbulence in the liquid phase has been developed by Lee (1987). The model can be described by a Stokes law based on an apparent turbulent viscosity of the fluid. It is expressed as:

$$C_D = \frac{2L}{\widetilde{\mathrm{Re}}_p}, \qquad \widetilde{\mathrm{Re}}_p = \frac{(u_{G0} - u_{L0})D_p}{\tilde{\nu}_L};$$

$$\frac{\tilde{\nu}_L}{\nu_L} = 100\,\epsilon_G^{0.5}\,\mathrm{Fr}^{-2.33}\,\mathrm{Re}^{0.86}\,\mathrm{St}^{0.3} \quad \text{for} \quad \widetilde{\mathrm{Re}}_p > 10;$$

$$\frac{\tilde{\nu}_L}{\nu_L} = 1 \quad \text{for} \quad \widetilde{\mathrm{Re}}_p < 1_0.$$

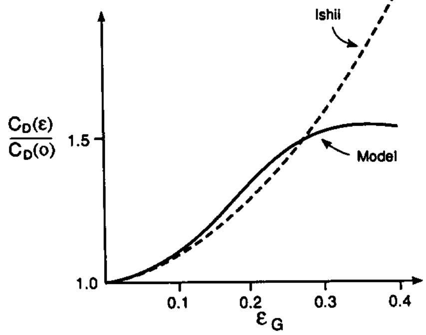

Figure 4. Comparison of the prediction in this paper with Ishii & Zuber's (1979) correlation.

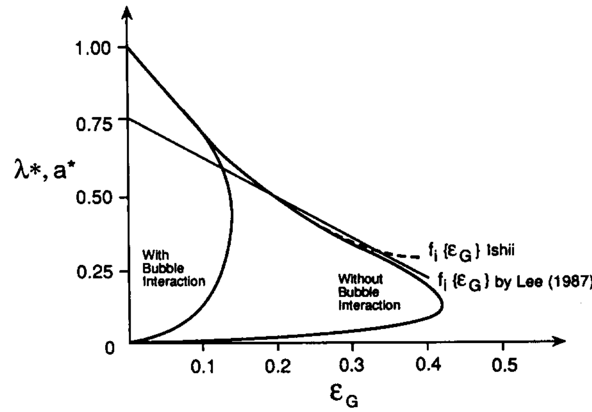

Figure 5. Effect of bubble interactions on the characteristic speeds. Kinematic wave velocities resulting from the models by Ishii & Zuber (1979) and Lee (1987).

Note that in this model, the dominant contribution to the drag is due to the void fraction. Including this drag coefficient in our model leads to the following kinematic velocity:

$$a_0^* = 0.75 - 1.25\epsilon_{G0}.$$

Figure 5 shows that in this case kinematic waves should be stable up to a void fraction of 0.4.

3.3. *Effect of wall shear stress*

We now go back to the linearized system [14a–c] and include the effect of wall shear stress:

$$\frac{\partial \epsilon_L}{\partial t} + \epsilon_{L0}\frac{\partial u_L}{\partial z} = 0 \qquad [24a]$$

and

$$2f_{i0}(u_G - u_L) - \epsilon_L \mathrm{Fr}_0 - 2f_{w0}u_{L0}\epsilon_{L0}u_L - f_{wL}\epsilon_0 u_{L0}^2 = 0. \qquad [24b]$$

At steady state,

$$f_{i0} - \epsilon_L u_{L0}^2 f_{w0} - \epsilon_{L0}\mathrm{Fr}_0 = 0.$$

The resulting kinematic wave is

$$a_o^* = (1 - \tfrac{3}{2}\epsilon_{G0})\left(1 - \frac{f_{wo}}{f_{i0}}u_{L0}\epsilon_{L0}\epsilon_{G0}\right). \qquad [25]$$

The wall friction term therefore introduces a small correction to the kinematic wave velocity when we compare [25] with [17]. Returning to the definition of f_i and f_w in [6]–[8], note that the correction is very small if the average bubble radius $R \ll D$, the pipe diameter. Except in unusual circumstances, the wall friction effect does not change the conclusions of the previous section.

3.4. *Effect of bubble interactions*

Bubble interactions, as they affect the virtual mass coefficient C_{VM}, the phasic pressure difference coefficient, ξ, and the velocity distribution coefficient k are now investigated. To bound the possibilities, consider bubbles in a cell where the influence of surrounding bubbles is integrated into a larger bubble encompassing a single bubble. This approach was used previously by Zuber (1964) to determine the virtual mass coefficient only. Although Ruggles *et al.* (1986) present measurements indicating that such a model overestimates interaction effects, nonetheless it is instructive to use this model as one extreme of what is possible.

The motion of the fluid bounded by two concentric spheres of radii a and b ($a < b$) moving at velocities u_G and u_L is given by Milne-Thompson (1968) (assuming the flow is inviscid and incompressible) as

$$\phi = \frac{\cos\theta}{c^3}\left[a^3 u_G - b^3 u_L)r + \frac{a^3 b^3}{2r^2}(u_G - u_L)\right], \qquad [26]$$

where

$$c^3 = b^3 - a^3\left(\epsilon_G = \frac{a^3}{b^3}\right).$$

Knowing the flow field, we can deduce the form Δp_{Li} and $\langle u_L'^2\rangle$. The result for C_{VM}, ξ and k is

$$C_{VM}(\epsilon_G) = \frac{2\epsilon_G + 1}{2\epsilon_L}, \qquad \xi(\epsilon_G) = \frac{1+\epsilon_G}{4\epsilon_L}, \qquad k(\epsilon_G) = \frac{1 + 5\epsilon_G\epsilon_L}{5\epsilon_L^2}. \qquad [27]$$

C_{VM} was reported by Zuber (1964). To our knowledge, the forms of $\xi(\epsilon_G)$ and $k(\epsilon_G)$ have not been published previously.

The model resulting from [27] can be written exactly as [10a,b] with the following new form for v:

$$v = \frac{\epsilon_L^2(C_{VM} - \xi - k\epsilon_G)^2}{\epsilon_G\epsilon_L + C_{VM}} + \epsilon_G\epsilon_L(\xi + k - C_{VM}) + 2\epsilon_L^2\left(\xi - \frac{C_{VM}}{2}\right) + \epsilon_L^2\epsilon_G\left(\frac{\partial\xi}{\partial\epsilon_L} + \epsilon_G\frac{\partial k}{\partial\epsilon_L}\right). \qquad [28]$$

The resulting characteristics are shown in figure 5 and compared with the noninteraction model. Note that the faster characteristic is not affected up to a void fraction of about 0.1, after which there is a rapid deviation. This interaction model predicts instability for $\epsilon_G > 0.12$ but probably overestimates the bubble interaction effect, as pointed out previously. Nonetheless the results of the analysis presented previously should hold at least for $\epsilon_G < 0.1$ and perhaps beyond on the basis of the empirical evidence.

3.5. *Qualitative effects of turbulence*

Just as the effects of velocity distributions around a single bubble were considered, we may also introduce the effects of turbulence through the velocity fluctuations about the mean. For simplicity, an eddy diffusivity model is used to model the Reynolds stress component of the axial velocity fluctuations, i.e.

$$\langle u_L'^2\rangle = D_T\frac{\partial u_L}{\partial z} + k\epsilon_G(u_G - u_L)^2, \qquad [29]$$

where $k = 1/5$ in dilute bubble mixtures, and D_T may be thought of as an axial diffusion coefficient. Equation [19] now becomes

$$\frac{\partial\epsilon_L}{\partial t} + a_0^*\frac{\partial\epsilon_L}{\partial z} = \left[\frac{\tau_0}{2f_{i0}}(\lambda^{*+} - a_0^*)(a_0^* - \lambda_0^{*-}) + a_0^*\frac{\epsilon_{G0}F_{T0}}{2f_{i0}}\right]\frac{\partial^2\epsilon_L}{\partial z^2}, \qquad [30]$$

where

$$F_{T0} = \frac{D_T}{(u_{G0} - u_{L0})D_{pipe}}$$

is called the dispersion number (Levenspiel 1968). Clearly turbulence tends to stabilize the flow. We see that the coefficient of the diffusive term can remain positive even when $a_0^* > \lambda^{*+}$ because of the turbulent term. This may explain experimental values of void propagation velocity slightly above λ^{*+}.

4. SUMMARY

It is shown that by accounting for the virtual mass force (given by [4]), phasic pressure difference (given by [3]) and liquid velocity distribution around a bubble (given by [5]), and neglecting bubble interactions, the bubble flow model based on the volume/ensemble- (or time-) averaged formulation has real characteristics up to a void fraction of ~ 0.4. Clearly at the higher void fractions the model is inaccurate due to bubble interactions.

The characteristic velocities, which represent the fastest and slowest propagating speeds in the system, are of the form

$$\lambda^{\pm} = u_L + (u_G - u_L) f^{\pm}(\epsilon_G) \quad \text{with} \quad 0 \leqslant f^{\pm}(\epsilon_G) \leqslant 1 \quad \text{and} \quad f^{+}(0) = 1, \quad f^{-}(0) = 0.$$

The Riemann invariants were determined and it was shown that the void fraction is approximately conserved along λ^{+}. The supports the interpretation of the experimental results on void fraction in Part I in terms of the characteristic velocities.

On the other hand, the void propagation velocity is also expected to be that of the kinematic waves. To clarify the situation the velocity of kinematic waves and their stability have been examined. Using the simplest interfacial drag correlation with a constant coefficient, the linear kinematic wave velocity is found to lie slightly outside the region encompassed by the faster and slower characteristics. This implies that they would be weakly amplified, as demonstrated by our analysis, though axial diffusion due to turbulence would tend to somewhat stabilize the situation. In fact, the experimental data on void propagation are seen to be between the faster characteristic velocity and the kinematic wave velocity derived with a constant drag coefficient. It was shown that inclusion of wall friction has negligible effect on these results.

If it is assumed that the kinematic waves are neutrally stable, i.e. their velocity is the same as the faster characteristic, then the functional dependence of the interfacial friction factor on the void fraction can be deduced. This relation agrees with the correlations proposed by Ishii & Zuber (1979) up to a void fraction of 0.3, but in fact the analysis can only be expected to be reasonable up to $\epsilon_G \sim 0.1$ to 0.15. This is still remarkable because no empirical input is required to derive the result except to assume that the inviscid terms, e.g. virtual mass term can be written in the dilute mixture limit. Indeed, the work of Ruggles *et al.* (1986) indicates that this assumption may not be as bad as we thought previously.

Nonetheless to clarify the effect of interactions, a bubble-in-cell model similar to Zuber's was used to calculate the characteristics. Though this model may significantly overestimate the effect of bubble interaction, still it was shown that the faster characteristic is unaffected up to a void fraction of 0.1. The results presented are therefore likely to be correct up to this value and probably beyond on the basis of experimental data. Furthermore, the effect of turbulence is shown to stabilize the void propagation equation by adding a diffusive component. Quantitative evaluation will not be attempted until more is understood about the axial diffusion coefficient.

Acknowledgement—This work was partially supported by the National Science Foundation under Grant CPE 82-12667.

REFERENCES

BANERJEE, S. & CHAN, A. M. C. 1980 Separated flow model, I. analysis of the averaged and local instantaneous formulation. *Int. J. Multiphase Flow* **6**, 1–24.

BERNIER, R. N. J. 1981 Unsteady two phase flow instrumentation and measurement. Report E 200-4, Division of Engineering and Applied Science, California Institute of Technology.

BIESHEUVEL, A. & VAN WIJNGAARDEN, L. 1984 Two-phase flow equations for a dilute dispersion of gas bubbles in liquid. *J. Fluid Mech.* **168**, 301–318.

DELHAYE, J. M. 1968 Equation fundamentales des ecoulements diphasiques. Report CEA-R-3429.

DREW, D. A. 1983 Mathematical modelling of two phase flow models and their applicability to nuclear reactor transient analysis. EPRI Report NP 163, Vols 1–3.

JONES, A. V. & PROSPERETTI, A. 1985 On the suitability of first order differential models for two phase flow prediction. *Int. J. Multiphase Flow* **11**, 133–168.

ISHII, M. 1975 *Thermo-fluid Dynamic Theory of Two Phase Flow*. Eyrolles, Paris.

ISHII, M. & ZUBER, N. 1979 Drag coefficient and relative velocity in bubbly, droplet or particulate flows. *AIChE Jl* **25**, 843–855.

LEE, S. L. 1987 Particle drag in a dilute turbulent two phase suspension flow. *Int. J. Multiphase Flow* **13**, 247–256.

LEVENSPIEL, O. 1968 *Chemical Reaction Engineering*. Wiley, New York.

MILNE-THOMPSON, L. M. 1968 *Theoretical Hydrodynamics*, 5th edn. Macmillan, New York.

PAUCHON, C. & BANERJEE, S. 1986 Interphase momentum interaction effects in the averaged multifield model. Part I: void propagation in bubbly flows. *Int. J. Multiphase Flow* **12**, 559–573.

RAMSHAW, J. D. & TRAPP, J. A. 1978 Characteristics, stability and short wavelength phenomena in two phase flow equation systems. *Nucl. Sci. Engng* **66**, 93–102.

RUGGLES, A. E., LAHEY, R. T. & SCARTON, H. A. 1986 An investigation of sound propagation in bubbly air–water flows. Paper presented at the *1986 Winter A. Mtg ASME*, Anaheim, Calif.

VOINOV, D. V. 1973 On the force acting on a sphere in a non-uniform flow of ideal incompressible fluid. *Zh. prikl. Mekh. Tekh* **4**, 182–184.

WALLIS, G. B. 1967 *One-dimensional Two-phase Flow*. McGraw-Hill, New York.

WHITHAM, G. B. 1974 *Linear and Nonlinear Waves*. Wiley Interscience, New York.

YADIGAROGLU, G. & LAHEY, R. T. 1976 On the various forms for the conservation equations in two phase flow. *Int. J. Multiphase Flow* **2**, 437–494.

ZUBER, N. 1964 On the dispersed two-phase flow in the laminar flow regimes. *Chem. Engng Sci.* **19**, 897.